Graphs and Networks

1. (2 points)

\[ \bar{D} = 6 \] (1pt)

\[ \bar{C} = \frac{8 + 4}{2} = \frac{0.286 + 0.667}{2} = 0.476 \] (1pt)

2. (4 points)

<table>
<thead>
<tr>
<th></th>
<th>Graph a</th>
<th>Graph b</th>
<th>Graph c</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree distribution</td>
<td><img src="degree.png" alt="Graph a" /></td>
<td><img src="degree.png" alt="Graph b" /></td>
<td><img src="degree.png" alt="Graph c" /></td>
</tr>
<tr>
<td>avg. degree</td>
<td>(\approx 2)</td>
<td>5</td>
<td>(\approx 2.67)</td>
</tr>
<tr>
<td>avg. path length</td>
<td>rel. low ((\approx 3.1))</td>
<td>low (1)</td>
<td>rel. high</td>
</tr>
<tr>
<td>clustering coefficient</td>
<td>very low (0)</td>
<td>very high (1)</td>
<td>rel. low</td>
</tr>
<tr>
<td>betweenness</td>
<td>1:high, 2:low</td>
<td>1,2: low</td>
<td>1:low, 2:high</td>
</tr>
</tbody>
</table>

3. (a) (2 points) By definition, each node has \(2k\) neighbors. The number of possible connections between these \(2k\) neighbors is

\[ N_{\text{possible}} = \frac{1}{2} \cdot 2k \cdot (2k - 1) = k \cdot (2k - 1) \]

since each of the \(2k\) neighbors can be connected to its \(2k - 1\) other neighbors. The factor \(\frac{1}{2}\) is added since the connections are not directed.

To count the number of actual connections, we consider each neighbor from left to right, counting only once each connection - i.e. counting only connections going to the right - see Figure 1. On the left side, each one of the \(k\) neighbors has \(k - 1\) connections. For the right side, the first neighbor has \(k - 1\) connections, the second \(k - 2\) connections, and so on.
Using the equivalence \( 1 + 2 + \ldots + n = \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \) we therefore have:

\[
N_{\text{actual}} = \left( k - 1 \right) + \left( k - 1 \right) + \ldots + \left( k - 1 \right) + \left( k - 1 \right) + \left( k - 2 \right) + \ldots + 1
\]

for the \( k \) left neighbors

\[
= k \cdot (k - 1) + \frac{1}{2}(k - 1)k
\]

\[
= \frac{3}{2}k^2 - \frac{3}{2}k = \frac{1}{2}k \cdot (3k - 3).
\]

The clustering coefficient is therefore:

\[
C = \frac{N_{\text{actual}}}{N_{\text{possible}}} = \frac{\frac{1}{2}k \cdot (3k - 3)}{k \cdot (2k - 1)} = \frac{3k - 3}{4k - 2} \quad \Box
\]

(b) (3 points) Let us assume a node \( v \) in the regular graph, where two of the neighbors — \( u \) and \( w \) — are connected by an edge \( e \). A total of three edges are involved in the case, as can be seen in the figure below:

During the rewiring, any of these edges can be rewired with probability \( p \), and therefore, are not rewired with probability \( 1 - p \). Edge \( e \) will only be a connection between neighbor nodes if none of the three original edges are rewired. This probability is expressed as \( (1 - p)^3 \). Thus, \( C(p) \) can expressed as:

\[
C(p) = C \cdot (1 - p)^3 = \frac{3k - 3}{4k - 2} (1 - p)^3
\]

which leads to:

\[
\frac{C(p)}{C(0)} = \frac{C \cdot (1 - p)^3}{C} = (1 - p)^3
\]
Dynamical Systems – Iterative Maps

4. (2 points)

<table>
<thead>
<tr>
<th>Attractor</th>
<th>Map a</th>
<th>Map b</th>
<th>Map c</th>
<th>Map d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value(s)</td>
<td>$x = 0.459$</td>
<td>-</td>
<td>$x = 0.636$</td>
<td>$x = 0.558/x = 0.765$</td>
</tr>
</tbody>
</table>

The solutions are considered correct if they are within $x \pm 0.05$.

5. (2 points + 1 bonus)

(a) Strange (or chaotic) attractor.

(b) $x_{t+1} = r \sin(\pi x_t) \rightarrow r = \frac{x_{t+1}}{\sin(\pi x_t)}$

$r = \frac{0.6}{\sin(\pi 0.6)} = 0.63088$

**Bonus** Any value $r = 0.72 \pm 0.05$ for the lower bound and $r = 0.83 \pm 0.05$ for the upper bound is accepted.

Fractals

6. (2 points + 1 bonus)

(a) This fractal – called *hexaflake* – can be constructed as follows: Start out with a hexagon, then in every iteration exchange each hexagon with a flake of 7 hexagons, each of them having a side length of $\frac{1}{3}$ of the former iteration.

(b) The Hausdorff dimension of this fractal is:

$$D = \frac{\log(7)}{\log(3)} \approx 1.7712$$

**Bonus** The hexaflake contains an infinite number of *Koch snowflakes*.

7. (3 points) The first two applications of the rule lead to the following strings:

0 F (Axiom)

1 FF[+F][−F]F

The corresponding geometric realization of the axiom and the first two applications of the rule are shown below: