Chapter 5: FUZZY Logic

(based on Lecture notes on “Real-world computing”, by Rolf Pfeifer)

“Crash course in Fuzzy Logic”

HS 2013
Logic so far:

propositional calculus

first order predicate calculus

limitations?

- cannot quantify over predicates
- only “statements” (not: I want; should be -- modal logics)
- only T or F whereas in the real world, everything is only true or false to a certain degree

--> Fuzzy Logic
Chapter 5: FUZZY Logic

Fuzzy sets: contrast to “crisp” (classic) sets

Fuzzy logic: automated reasoning/decision making with fuzzy sets
Chapter 5: FUZZY Logic

Fuzzy sets: contrast to “crisp” (classic) sets
historically: Lotfi Zadeh, Berkeley, 1965
Modern version: Bart Kosko

Fuzzy logic: reasoning/decision making with fuzzy sets --> BIOFAMS (Binary Input-Output Fuzzy Associate Memories)
Fuzziness vs. randomness

“Hold an apple in your hand. Is it an apple? Yes. The object in your hand belongs to the clumps of space-time we call the set of apples — all apples anywhere, ever. Now take a bite, chew it, swallow it. Let your digestive tract take apart the apple’s molecules. Is the object in your hand still an apple? Yes or no? Take another bite. Is the new object still an apple? Take another bite, and so on down to void.” (Kosko, 1992, p. 4)

Initially the apple is clearly an apple. But as the number of pieces bitten off increases, it gradually loses the property of “apple-ness”. At the end, when the apple has been completely eaten, it is no longer a member of the class of apples. The basic idea of fuzzy logic is to associate a number with each object indicating the degree to which it belongs to a particular class of objects. Initially, for your apple, this number will be 1 or close to 1. At the end it will be zero, since the apple ceases to exist. In between it will be slowly decreasing. The function that associates a number with the object is called the membership function. In classical set theory this function is either 1 (the object belongs to the set) or 0 (the object does not belong to the set); it is also called the “characteristic” function.

Other example:
- circle drawn by hand
- there is a 20% chance of light rain tomorrow
(difference between probability and fuzzyness).
Fuzzy sets - definitions

classical “crisp” set: characteristic function \{0,1\}
fuzzy set: characteristic function \(m(A)\): between 0 and 1

"comfortable house for a four-person family"
- cardinality
- intersection
- union
- complement
- degree of fuzziness \(E(A)\)
A car can be viewed as "domestic" or "foreign" (from an US-centric view) from different perspectives. One is that a car is domestic if it carries the name of a US manufacturer, otherwise it is foreign. However, the distinction is not as crisp. Many domestic cars are produced outside the US. Or we can look at the number of parts manufactured abroad, etc. Figure 7.2 provides examples of membership functions.
Fuzzy sets - definitions

cf script

- cardinality
- intersection
- union
- complement
- degree of fuzziness (how fuzzy is a fuzzy set?)

p. 105 bottom
"comfortable house for a four-person family"
- cardinality
- intersection
- union
- complement
- degree of fuzziness E(A)
The fuzzy subset A is a point in the 2-dim unit cube with coordinates (1/3 3/4). The first element of A, x1 fits in or belongs to A to degree 1/3, the second, x2, to degree 3/4. The cube consists of all possible fuzzy subsets of two elements \{x1, x2\}. The four corners represent the power set of the classical set, consisting of 2 elements \{x1, x2\} (from Kosko, 1992, p. 270).

Kosko Cube: Fuzzy power set F(2**X) - set of all subsets.

Vertices (corners) in cube define nonfuzzy sets.
for only two elements \{x1,x2\} the power set is \{\emptyset, \{x1\},\{x2\},\{x1,x2\}\}. These four sets correspond to the four bit vectors (0,0), (1,1),(1,0), and (0,1). (see p. 108)
Operations on fuzzy sets: examples

definitions: p. 108 bottom
examples: p. 109 top, sets A and B
Fuzzy set operations in Kosko Cube
Interesting considerations at midpoint. Kosko argues that classical logic excludes this midpoint. At midpoint nothing is distinguishable, there are no contradictions. Elimination of this points leads to the paradoxes of which there is a lot in classical logic (like the liar from Crete who said that all Cretans are liars). Moreover, Kosko argues, that the paradoxes are the result of Western yes/no thinking, whereas Eastern Yin-Yang thinking has no problems with this.
If we ignore $x_1$ for a moment, we see that $A$ is indeed a subset of $B$. But even if we include $x_1$, $A$ is still almost a subset of $B$. In other words, there is something like a degree of subsethood. Let us call it $S(A,B)$. It is high in this example. Since there is only one violation, i.e. one position in which $A$ is not subsumed by $B$, the degree of subsethood is large. Thus, for large sets (i.e. $M(A)$ large) and only few violations, subsethood is high. The more violations, the less $A$ is a subset of $B$, but the more $A$ is a superset of $B$. What we see here is that to some degree one set is subset of the other, but at the same time it is to some degree a superset of the other.
Calculating Fuzzy subsethood
(see script, p. 112)

\[ S(A, B) = 1 - \frac{\sum_{x \in X} \max(0, m_A(x) - m_B(x))}{M(A)} \]

If \( m_A(x) - m_B(x) \) is always \( \leq 0 \), the numerator will always be zero. This implies that \( A \) is entirely a subset of \( B \) (Zadeh's definition). In other words, set \( A \) in the cube between \( B \) and the origin (see figure).
Calculating Fuzzy subsethood

(see script, p. 112)

\[ S(A, B) = 1 - \frac{\sum_{x \in X} \max(0, m_A(x) - m_B(x))}{M(A)} \]

The point shown outside the shaded rectangle is subset as well as superset of B.
Fuzzy Logic

reasoning with Fuzzy sets
Fuzzy associative memories

Figure 7.9: FAM system architecture. Details: See text.
Fuzzy associative memories: control of a traffic light

“If traffic is heavy in this direction, then keep the light green longer”

How can we formalize this intuitively correct sentence?
Traffic density

Figure 7.10: The membership functions for HEAVY, MEDIUM, and LIGHT traffic. On the horizontal axis there is a measure of traffic density.
Fuzzy associative memories: control of a traffic light

“If traffic is heavy in this direction, then keep the light green longer”

fuzzy association: \((\text{HEAVY}, \text{LONGER})\)

input fuzzy variable \textit{traffic density}: value \text{HEAVY}

output fuzzy variable \textit{green light duration}: \text{LONGER}

additional fuzzy association: \((\text{LIGHT}, \text{SHORTER})\)

Figure 7.10: The membership functions for \text{HEAVY}, \text{MEDIUM}, and \text{LIGHT} traffic. On the horizontal axis there is a measure of traffic density.
BIOFAMS (Binary input-output fuzzy associative memories)

1. (p. 115; on white board)
BIOFAMS (Binary input-output fuzzy associative memories)

1. Identification of essential (linguistic) variables
2. Definition of membership functions (+labels)
3. Construction of a set of fuzzy rules (the FAM rules)
4. Combination of all outputs of the “rule set” into a geometrical output function
5. “Defuzzification” in order to get a non-fuzzy output value
The pole balancing problem

goal: adjust motor output \( v \) to balance inverted pendulum using a BIOFAM

Proceed through steps 1 to 5.
1. \( \Theta \) (angle between pendulum and vertical), \( \Delta \Theta \) (angular velocity)
   \[ \Delta \Theta = \Theta(i) - \Theta(i-1) \]
2. membership functions for \( \Theta \), \( \Delta \Theta \), \( v \)
   Designer’s choice: fine-grainedness
   NL: Negative Large
   NM: Negative Medium
   NS: Negative Small
   ZE: Zero
   PS: Positive Small
   PM: Positive Medium
   PL: Positive Large
   (linguistic variables)
Typical choice of membership functions

Proceed through steps 1 to 5.
1. $\Theta$ (angle between pendulum and vertical), $\Delta \Theta$ (angular velocity)
   
   $\Delta \Theta = \Theta(i) - \Theta(i-1)$

2. membership functions for $\Theta$, $\Delta \Theta$, $v$
   Designer’s choice: fine-grainedness
   
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   PM: Positive Medium
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   (linguistic variables)

   $\rightarrow$ FUZZIFICATION

3. Construction of FAM rules
Typical choice of membership functions

IF angle is negative medium AND the angular velocity is about zero THEN the motor velocity should be positive medium

IF Θ=NM AND ΔΘ=ZE THEN v=PM

(NM,ZE;PM)

Proceed through steps 1 to 5.
1. Θ (angle between pendulum and vertical), ΔΘ (angular velocity)
ΔΘ= Θ(i)-Θ(i-1)
2. membership functions for Θ, ΔΘ, v
Designer’s choice: fine-grainedness
NL: Negative Large
NM: Negative Medium
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PS: Positive Small
PM: Positive Medium
PL: Positive Large
(linguistic variables)
→ FUZZIFICATION
3. Construction of FAM rules
Rule bank for inverted pendulum

Figure 7.13: A FAM rule bank for the inverted pendulum problem.

Fill in interactively:
(a) use engineering or common sense knowledge
Why are some positions empty? ("taking care of themselves", "lost causes")
(b) sensitivity analysis: which rules are really required?
(c) apply adaptive procedures which are based on empirical data
### Rule bank for inverted pendulum

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Figure 7.13: A FAM rule bank for the inverted pendulum problem.

(a) use engineering or commonsense knowledge
Why are some positions empty? (“taking care of themselves”, “lost causes”)
(b) sensitivity analysis: which rules are really required?
(c) apply adaptive procedures which are based on empirical data
Correlation-minimum fuzzy inference procedure.
All rules are activated in parallel, but to varying degrees.
The degree depends on how well the input value matches the membership function on the left-hand side (input side, antecedent side).
The first rule is not applied since the input values $\Theta = 15$ and $\Delta \Theta = -10$ do not intersect any of their membership functions.
Next rule: the membership function for PS is intersected by $\Theta = 15$ at 0.8, the one for ZE is intersected by $\Delta \Theta = -10$ at 0.5.
These values are combined by using the minimum (logically speaking the AND) function: $\min(0.8, 0.5) = 0.5$. Thus, this rule is applied to degree 0.5.
This degree of application of the rule is propagated to the output side as shown geometrically. Similarly for the third rule.
Now take maximum over all consequent sides of the rules (shaded areas): get fuzzy centroid at bottom. This is the geometrical output function that we are looking for. This is a kind of min-max procedure.
Now we need a non-fuzzy output value because the motors require one single crisp value for current. This is done by a “defuzzification” procedure.
Defuzzification

general formula:

\[
\bar{v} = \frac{\int v \cdot m_{\text{output}}(v) \, dv}{\int m_{\text{output}}(v) \, dv}
\]

in case of trapezoid: simplifications

5. Defuzzification: get crisp value from “centroid”. Calculate “center of gravity”.
Adaptive BIOFAM clustering

In control applications, humans or automatic controllers generate a continuous stream of obviously appropriate input-output data. Adaptive BIOFAM clustering converts this data to weighted FAM rules. In essence, BIOFAM clustering counts quantization vectors in FAM cells. The clustering procedure samples the nonfuzzy input-output stream \((x_1, y_1)(x_2, y_2), \ldots\) An unsupervised clustering procedure distributes the \(k\) quantization vectors \(m_1, \ldots m_k\) in \(XxY\). Learning distributes them to different cells in the rule bank. The key ideas is that each cluster equals a rule.

Procedure to perform adaptive BIOFAM clustering:
- identify the variables
- find the linguistic labels and define the membership functions
- take an existing, functioning controller, human or machine
- choose the number of quantization vectors (e.g. one for each cell)
- generate cases - for each input vector generate the appropriate output action (the control action). Register into which cells the output actions fall.
- perform sensitivity analysis. This may enable you to remove some of the rules, thus making the system smaller and more efficient.
- deal with the cases for which you do not have data on the basis of intuition or prior knowledge.
The “truck backer upper”

we just indicate the x-coordinate rather than the x-y coordinate. Input variables: x coordinate and the angle of the truck $\Phi$. The output variable is the steering angle $\Theta$. The ranges of the variables are indicated above.

The more challenging problem - not discussed here - will be parking the truck with a trailer.
The “truck backer upper”

Figure 7.18: The membership functions for the truck backer-upper.

Membership functions with different levels of resolution: high resolution near the center (CE), the vertical (VE), and zero (ZE), low resolution toward the edge. The corresponding rule bank is also shown.
Summary

• Fuzzy Logic successful in many applications
• takes intrinsic uncertainty of real world into account
• fundamental difference between probabilistic and fuzzy logic thinking
• fuzzy sets differ from classical crisp sets by their membership function
• intersection of fuzzy set with complement is not empty set
• “Kosko cube” for geometric visualization of fuzzy sets (points in n-dimensional hypercube
• five steps in development of fuzzy rule system
• fuzzy inference procedure apply all rules in parallel to varying degrees